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SENSIBILITY ANALYSIS USING POLYNOMIAL CHAOS: APPLICATIONS

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ABSTRACT
In this paper, we present robustness analysis of Control Systems by means of the Polynomial Chaos Method (PCM). The PCM was firstly conceived by Norbert Wiener to solve some kind of Gaussian stochastic system. The method was improved and generalized to be applied to different kinds of random measures (the Winer-Askey scheme). It consists of expanding candidate solutions in Galerkin polynomials of random variables, which transforms the system in a bigger set off deterministic equations that can be solved by traditional numerical methods. We also present two examples of applications: 1) robustness analysis of a multivariable linear time invariant system under a robust controller and 2) the robustness analysis of a nonlinear system with a feedback linearization controller.

INTRODUCTION
The problem of numerical solution of a stochastic differential systems is presented in more detail in [1], as presented in the following. The stochastic differential system to be simulated is:

$$\begin{cases} \dot{x}(t) = F(x(t), w(t), \theta, c, t) \\ v = H(x(T), w(T), \theta, c, t) \end{cases} \quad (1)$$

where
1. θ is a vector of random parameters;
2. $w(t)$ is a vector of input stochastic processes;
3. c is a vector of initial conditions (that are also random variables);

We are interested in random variables, that we could call v, in the final instant of time T, and we wish to determine the following parameters for this variable:

1. *Expected Value and Variance / Standard deviation;*
2. *Probability Density Function (PDF).*

The vector of parameters θ, initial conditions c and stochastic input stochastic processes $w(t)$ are all defined in the probability space $(\Omega; \mathcal{F};\ P)$. Despite the stochastic processes w_i be a uncountable set of random variables, in order to perform numerical integration, they must be approximately expressed as a finite set of random variables i.i.d. (independent and identically distributed (for example, by means of a Karhunen-Loeve expansion).
So, for all the purposes, the total randomness of the system are represented by a vector of n random variables:
$\Xi = [\xi_1\ \ \xi_2\ \ \cdots\ \ \xi_n]$

Definition 1.1 (Functional Solution). *The stochastic process x, that is solution of the problem presented in Eq. (1) is a functional of the random variables and input stochastic processes:*

$$x_t\ =\ x(t;\theta;w_t)$$

In many applications, w_t is a vector of Wiener processes, as inputs are affected by white noise.
The process of efficiently combining non-intrusive polynomial chaos expansions based methods uncertainty quantification methods and adjoint techniques to obtain robust optimal controllers for dynamical systems is present in [2, 3].

In [4] was exploited the polynomial chaos approximation to represent the residual energy random variable, was presented a three mass-spring system used to illustrate how the proposed approach can be easily extended to systems
With multiple uncorrelated uncertainties. Minimizing the expected value of the mean, variance and absolute value of the skew is shown to progressively approach the solution of the minimax design. A three-dimensional puff-based model has been tested in [5], for the purpose of accurately estimating the uncertainty distribution of the solution, caused by the uncertainty in the diffusion parameters of the model. The approximate solution to the

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stochastic system was obtained as a linear combination of the selected orthogonal basis functionals, whose coefficients are functions of time. The solution is shown to compare well with the true Monte Carlo solution.

In [6] is developed a novel framework for stability analysis of linear and polynomial stochastic systems. The developed
Theory is also applied for robustness verification of a linear flight control design for a stochastic F-16 aircraft model.
In this paper, will be presented robustness analysis of Control Systems by means of the Polynomial Chaos Method (PCM). With objective the goal of this work are the applications of control systems by means of the PCM applied in two examples: 1) robustness analysis of a multivariable linear time invariant system under a robust controller and 2) the robustness analysis of a nonlinear system with a feedback linearization controller.

POLYNOMIAL CHAOS METHOD

Series expansions of stochastic processes is a well-known fact that helps in the numerical solution of stochastic differential equations. In fact, a very general stochastic process, as a Wiener process, from which it is possible to define rigorously *white noise*, can be represented as a measure space $\mathcal{W}(\mathbb{R})$ with a σ-algebra of cylinders (the cylinders can be constructed from the neighborhoods of each possible trajectory of the Wiener process) [7]. In fact, given the Wiener probability space $\mathcal{W}(\mathbb{R})$, the Wiener measure ϖ is the only probability measure in $\mathcal{W}(\mathbb{R})$ such that when restricted to a finite subset of sections of the cylinders, for example $A_1;\ \cdots\ ; A_N$, results in:

$$P(A_1, \cdots, A_N) = \int_{A\_1} \cdots \int_{AN} fw(t_1, w_1) fw(t_2 - t_1, w_1, w_2) fw(t_3 - t_2, w_2, w_3) \cdots fw(t_n - t_{n-1}, w_{n-1}, w_n) dw_1 dw_2 \cdots dw_n \quad (2)$$

where fw is the Gaussian probability distribution. The set of random variables associated to this space are the elements of the Banach space $L_p(\mathcal{W}(\mathbb{R}))$, which are functionals in this space and are represented by $F[w]$, where $w \in \mathcal{W}(\mathbb{R})$. For any functional in this space, it is valid that:

$$\int_{\mathcal{W}(\mathbb{R})} |F[x]|^p d\varpi < \infty$$

Given the Hilbert space of squared integrable time functions $L_2(\mathbb{R})$, and the set of orthogonal functions $\{\alpha_n(t)\}$ that generates $L_2(\mathbb{R})$, the Karhunen-Loeve (KL) series for the Wiener process is an Fourier-like series expansion in those functions with coefficients given by the generalized Stieltjes integrals:

$$a_n = \int_a^b \alpha_n(t) dw(t)$$

where $w \in \mathcal{W}(\mathbb{R})$. Those integrals are of course random variables. Evidently, for more regular stochastic processes,
similar expansions exists.
Another kind of series expansion of a stochastic process, that is equally useful in the solution of stochastic differential
equations, is the expansion in functions of random variables. In particular, the Polynomial Chaos Expansion has several numerical advantages that justify its increasing popularity. It was Norbert Wiener, in [8], which shown that an Gaussian stochastic process could be expanded in a finite variance (that is a process in the space $L^2(\Omega; F;\ P)$) Fourier-Hermite series in Gaussian random variables (that is, the elements of the basis are Hermite polynomials in the random variables). In [9], Cameron and Martin have shown that the series converge in the L_2 sense [10].
Those random polynomials are also known as *Wick polynomials*. In fact, given a probability space $(\Omega; F;\ P)$ and a set of random variables $\Delta_1, \Delta_2, \cdots, \Delta_n$, the Wick product for the set, represented by: $\Delta_1^{k_1}, \Delta_2^{k_2}, \cdots, \Delta_n^{k_n}$: is given by:

1. $E[: \Delta_1^{k_1} \Delta_2^{k_2} \cdots \Delta_n^{k_n} :] = 0$
2. $\frac{\partial}{\partial \Delta_i}\ :\ \Delta_1^{k_1} \Delta_2^{k_2} \cdots \Delta_i^{k_i} \cdots \Delta_n^{k_n}\ := k_i\ :\ \Delta_1^{k_1} \Delta_2^{k_2} \cdots \Delta_i^{k_{i-1}} \cdots \Delta_n^{k_n}\ :$

For example, we have $: \Delta := \Delta - E[\Delta]$. One defines the *Wick exponential* from the Wick products as:

$$: e^{\alpha\Delta} := \sum_{i=0}^{\infty} \frac{\alpha^n}{n!} : \Delta^i :$$

The Wick powers $: \Delta^n :$ have the same formula as the Hermite polynomials. In the particular case of Δ be W_t, that is, the Wiener process in the instant of time t, the above formula is equivalent to the exponential $e^{\lambda t}$ of Ito Calculus [7]. In the case of several random variables $\Delta_1, \Delta_2, \cdots, \Delta_n$, we have the Wick polynomials depending on those variables. In the particular case of Δ being a Gaussian random variable of type $N(0, 1)$, we have:

$$: e^{\alpha\Delta} := e^{\alpha\Delta - \frac{1}{2}\alpha^2} = \Psi(\alpha, \Delta) \quad (3)$$

where $\Psi(x, z)$ is a generating functions of Hermite polynomials. The Hermite polynomials (not normalized) are obtained by the formula: $N_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$, with $n \in \mathbb{Z}_+$ that satisfy the following relation (see, for example [11]):

$$\int_{-\infty}^{\infty} e^{-x^2} N_m(m) N_n(x) dx = \begin{cases} 0 & m \neq n \\ 2^n n! \sqrt{\pi} & m = n \end{cases} \quad (4)$$

If the inner product in the Hilbert space of functions $L_2(\mathbb{R})$ is de_ned by the integral in Eq. (4), where the weight function is $w(x) = e^{-x^2}$, the the functions $N_n(x)$ form an orthogonal basis in this space. Some examples of Hermite polynomials are $N_0(x) = 1, N_1(x) = x, N_2(x) = x^2 - 1, N_3(x) = x^3 - 3x$. An orthonormal basis is created if the Hermite polynomials are normalized in the following way:

$$H_n(x) = \frac{N_n(x)}{\sqrt{n!}}$$

The normalized Hermite polynomials are related to the generating function presented in Eq. (3) by the following formula

$$\Psi(x, z) = e^{-\frac{x^2}{2} + xz} = \sum_{n=0}^{\infty} \frac{N_n(x)}{n!} z^n$$

There are also recursive formulas that can be used to calculated the set of normalized Hermite polynomials. The Hermite polynomials are related by the recursive formula:

$$\sqrt{n+1} H_{n+1}(x) - x H_n(x) + \sqrt{n} H_{n-1}(x) = 0$$

So, given a Gaussian random variable Δ, any function of Δ can be written as a Fourier-Hermite series:

$$f(\Delta) = \sum_{n=0}^{\infty} f_n H_n(\Delta)$$

where the coefficients can be calculated by formula:

$$f_n = \int_{-\infty}^{\infty} f(x) H_n(x) \rho(x) dx \quad (5)$$

It is important to emphasize here that the calculations in Eq. (5) are deterministic, that is, it is a normal integral calculation, where

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Given the random variable function $f(x)$, the following formula relates its variance with the Fourier-Hermite coefficients:

$$E[f^2(x)] = \sum_{n=0}^{\infty} |f_n|^2 \quad (6)$$

It is easy to show that $E(N_0(\Delta)) = 1$ and $E(N_n(\Delta))$.

Solution of Stochastic Differential Systems using Polynomial Chaos Method

The Polynomial Chaos method can be used to solve a stochastic differential system (SDS), that could be ordinary or partial. Using the result presented in [9], the solution of the SDS is of the form:

$$u(x, t; \xi_1, \xi_2, \cdots, \xi_n) = \Gamma_0(x, t) + \sum_{i=1}^{n} \Gamma_i(x, t, \xi_i) + \sum_{i=1}^{n-1} \sum_{i<j}^{n} \Gamma_{ij}(x, t, \xi_i, \xi_j) + \cdots \quad (7)$$

where $\xi_1, \xi_2, \cdots, \xi_n$ is a set of random variables such as all the random parameters and inputs can be expressed in function of them (by means, for example, of Karhunen-Loeve series). In particular:

1. $\Gamma_0(x, t)$ is the expected value of the stochastic process;
2. $\Gamma_i(x, t) = E[u_{x,t} | \xi_i] - \Gamma_0(x, t)$

For the particular case of an ordinary stochastic system with one random parameter θ, the general form [1] is:

$$\dot{u}_t = F(u_t, w_t, \theta, t) \quad (8)$$

where t is the time variables, $\theta(\omega)$ is the random parameters in the probability space $(\Omega;\ F;\ P)$, $w_t(\omega)$ is a vector of stochastic process that represents inputs to the system (in function of θ), and $c(\omega)$ is a vector of initial conditions (in function of θ). The general solution is of the form:

$$u(t; \omega) = \sum_{i=0}^{\infty} u_i(t) \phi_i(\omega) \quad (9)$$

where $\{\phi_i(\omega)\}$ is a basis for the Hilbert space $L^2(\Omega; \mathcal{F};\ P)$, that are random variables in function of θ (so are dependent variables). The functions/coefficients $\{u_i(t)\}$ are then deterministic functions. In particular, the solution of the system in Eq. (8) must have such a decomposition, and $\phi_i(\omega)$ must be functionals in $L_2(\Omega; \mathcal{F};\ P)$. If the solution is numerical, as in the vast majorities of the cases, some statistical parameters are commonly calculated, as for example means, standard deviations and estimates for PDF (probability density functions). In particular,

$$Var[u(t, \omega)] = E[u^2] - (E[u])^2 = \sum_{i=0}^{\infty} u_i^2\, E[\phi_i^2(\Delta)] - u_0^2$$

RESULTS AND DISCUSSION

Application to a Linear Control System

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In this section, we present a robustness analysis of a linear time invariant system that represents a multivariable plant.
The plant/process controlled in this work is the kit LTR 701 from the manufactures Amira/Elwe, that is in the Control Laboratory at the Sorocaba Campus - São Paulo State University (UNESP).

A picture of the plant is in Fig. 1. This kit consists of an air conducting pipe, in which the air is forced by a proportionally controlled fan (which dynamic is deliberately ignored). The fan's velocity is varied between 0 to 100 per cent, which corresponds to 0 to 10 V in the analog input signal. A heating resistance is right after the fan, which also can be varied from 0 to 100 per cent. There is a manual buttery entrance valve, right before the fan, that can be used to generate disturbances in the system. There is a thermo couple sensor to measure the air temperature, which can be put in four different places along the tube. At the other end of the tube, an air mass flux sensor or a pressure sensor can be connected. Those signals are internally converted in analog voltage signals, which are also accessible in the front panel (or in a connector to the computer system to be used in the control function). The acquisition board is the MF-614 from Humusoft, which has seven analog inputs and two analog outputs (the digital inputs and outputs will not be used in this work). The analog inputs can be configured to accept signal from -10 to 10 V, which in compatible to the voltage levels of the kit. The software used for control / system identification used is a real time software (real time Windows target, that is a toolbox of MATLAB/Simulink). This tool generates a simulation / control executable file that runs with higher priority than the operational system (kernel) and takes control of the hardware (and of the interrupt request management), guaranteeing the hard real time demands. The thermo couple sensors are of the type NiCr-Ni, with time lag constants of 0.3 or 3.0 seconds, depending on which sensor is used (this constants will be ignored). The more distant the sensor is put away from the heating resistance, the greater is the dead time in the corresponding transfer function. The linear system to be controlled is shown in Fig. 1 below.

Fig. 1. Multivariable System to be Analyzed

$$A=\begin{bmatrix} -47.48 & -17.57 & -2.28 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5.92 & -2.01 & -0.76 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -25.80 & -11.63 & -5.29 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16.00 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix};\ B=\begin{bmatrix} 0.50 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.25 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 8.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix};\ C=\begin{bmatrix} 0.50 & -0.06 & 0.62 & 0.00 & -0.043 & 0.34 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -4.01 & 8.36 \end{bmatrix};\ D=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (10)$$

For the case of the control system analyzed in this work, some parameters of the matrix A presented in Eq. (10) were varied, that is, were considered random variables with uniform distribution with 10% of variation around the nominal value. For this kind of PDF, the corresponding polynomials are Legendre polynomials. The number

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of polynomials in the truncated base were ten, so the corresponding deterministic system of differential equations has ninety equations. In Fig. 2, it is shown the map of eigenvalues for the nominal system (in green - the transmission zeros are also presented) and the stochastic system (in magenta - without transmission zeros). All the eigenvalues has some variation with this parameter, but the two leftmost has a greater variation.

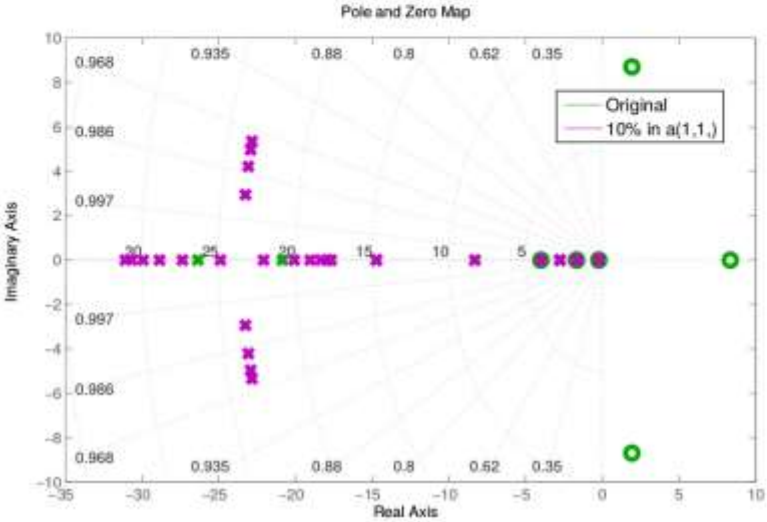

Fig. 2. 10% variation in A(1, 1)

In Fig. 3. In Fig. 4, on the other hand, are shown the poles for the closed loop system with $\mu = 0.05$ in the LQG/LTR design. It is clear that the poles' dispersion are considerably lower near the imaginary axis (there are poles in the left that are not shown - the zeros should be ignored in this figure). Of course, the controller changed the positions of some poles, besides the reduction of dispersion. The same occur with variations in $A(1,1)$ in closed-loop.

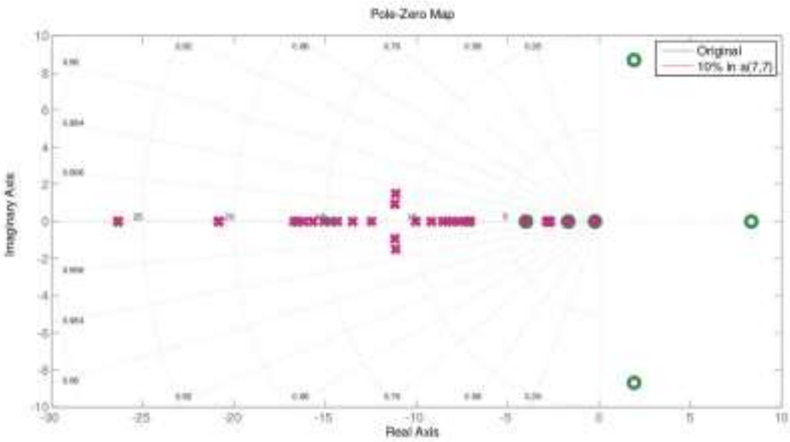

Fig. 3. 10% variation in A(7, 7)

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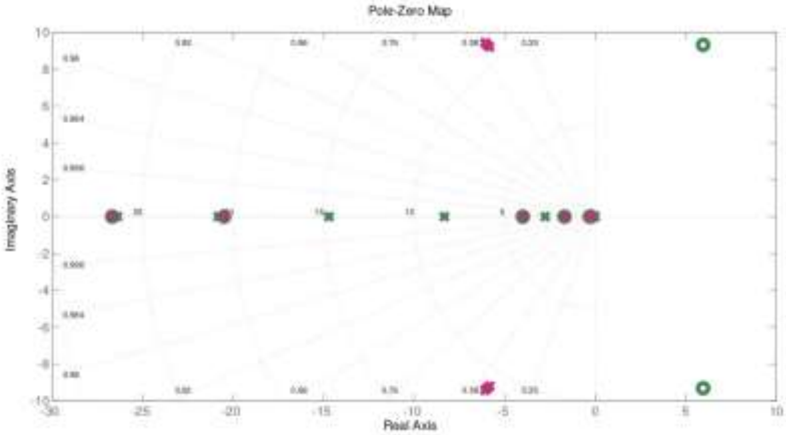

Fig. 4. 10% variation in A(7, 7) and closed-loop

Application to a Nonlinear Control System: Feedback Linearization

The Duffing oscillator is a very common kind of nonlinear system that appears in several applications, that is, many nonlinear systems can be modeled by this equations. It could represent, for example, a mass-spring-damper system where the spring has a cubic dependence on the position. The differential equation that represents this system is

$$\ddot{y} + 2\xi\dot{y} + \alpha y + \gamma y^3 = u(t) \tag{11}$$

where y is the position, ξ is the damping coefficient, α is the linear spring constant and is the cubic spring constant. The input signal $u(t)$ could also be a feedback signal.

In order to apply the feedback linearization procedure, one has to put the system in the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u(t) \\ y &= h(x) \end{aligned} \tag{12}$$

which is a second order system with vector fields:

$$f(x) = \begin{bmatrix} x_2 \\ -\alpha x_1 \quad -2\xi x_2 \quad -\gamma x_1{}^\wedge 3 \end{bmatrix} \tag{13}$$

$$g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{14}$$

Let us suppose that we could choose a linear combination of the states as output. Then:

$$h(x) = [a \quad b] \tag{15}$$

After applying time derivative in the output $z = h(x)$, the input appears, and the relative degree is one. Then we have:

$$\dot{z} = (a + 2b\xi)x_2 - b\alpha x_1 - b\gamma x_1^3 + bu$$

In order to proceed the linearization, one must find a diffeomorphism Φ that transforms the coordinates x_1, x_2 to new coordinates μ, ψ where $\mu = y$, and ψ is determined by the following condition:

$$\nabla\psi \cdot g = 0$$

which guarantees that ? that the system in the new coordinates is separated in observable and non-observable parts ?. Such condition could be achieved if $\psi = cx_1$. Then, the inverse diffeomorphism is simply:

$$\Phi^{-1}(\mu, \psi) = \left(\frac{1}{c}\psi, \frac{1}{b}\mu - \frac{a}{bc}\psi\right)$$

The original system in the new coordinates and e with the linearizing control law

$$u(t) = \frac{1}{b}\left\{\left(-\frac{a}{b} + 2\xi\right)\mu + \frac{1}{c}\left(\frac{a^2}{b} + ab - 2\xi a\right)\psi + \frac{\gamma b}{c^3}\psi^3 + v\right\} \tag{16}$$

is given by:

$$\begin{aligned} \dot{\mu} &= v \\ \dot{\psi} &= \frac{b}{c}\mu - \frac{a}{b}\psi \\ y &= \mu \end{aligned} \tag{17}$$

In order to find the zero dynamics, one imposes $\mu = 0$, which shows that the closed loop system is stable if a and b have the same sign.

CONCLUSION

In this paper, we present robustness analysis of Control Systems by means of the Polynomial Chaos Method (PCM). We also present two examples of applications: 1) robustness analysis of a multivariable linear time invariant system under a robust controller and 2) the robustness analysis of a nonlinear system with a feedback linearization controller.

The integration of such system produces some special realizations of the solution stochastic process. All the other realizations can be calculated from those. Some observations follows:

1. The resulting set of deterministic differential equations can be solved by traditional numerical methods;
2. The method can only be applied in polynomial systems, that is, the state variable appears in polynomial form in the stochastic differential equations, as well as the random parameters. Eventual transcendental functions must be approximated by finite sum of polynomials, as for example Taylor polynomials;
3. The number of terms in the expanded (deterministic) equations can grow very fast with the number of polynomials in the truncated expansion;
4. In contrast with the Monte Carlo method, pseudorandom numbers are only needed after the differential system solution, and in case we want to estimate PDF (probability density functions).

Applications to closed-loop control systems are also possible. In this case, the input to the systems must also be a stochastic process, which will depend on the random parameters as well. It could be used for example to robustness analysis. [12].

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